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Recurrent construction of q-boson realizations of quantum matrix element algebras of quantum group $GL(n)_q$

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Abstract. The q-boson realizations of the quantum matrix element algebra $A(n)_q$ of the quantum group $GL(n)_q$ are recurrently constructed. Generators of $A(n)_q$ are expressed as $n-1$ independent q-boson operators and the generators of $A(n-1)_q$.

Quantum groups are the mathematical structure specified from quantum R-matrix satisfying the quantum Yang–Baxter equations through the FRT procedure [1, 2]. It can also be explained as a transformation group on a quantum plane [3]. The quantum group $GL(n)_q$ [4] is a set of matrices M, with non-commuting elements m_{ij} satisfying

$$\begin{aligned} &m_{ij}m_{ik}=q^{-1}m_{ik}m_{ij} && j<k \\ &m_{ij}m_{kj}=q^{-1}m_{kj}m_{ij} && i<k \\ &m_{ij}m_{kl}=m_{kl}m_{ij} && i<k \text{ and } j>l \\ &m_{ij}m_{kl}=m_{kl}m_{ij}+(q^{-1}-q)m_{il}m_{kj} && i<k \text{ and } j<l \end{aligned} \tag{1}$$

and with the quantum determinant $D_q(M)$

$$D_q(M)=\sum_{s\in S_n}(-q)^{-l(s)}m_{1s_1}m_{2s_2}\ldots m_{ns_n} \tag{2}$$

where S_n is symmetric group and $l(s)$ is the minimal number of permutations in s, not vanishing. The quantum matrix element algebra $A(n)_q$ of $GL(n)_q$ is an associative algebra over $\mathbb{C}$ generated by m_{ij} satisfying relations (1).

Using their Heisenberg–Weyl relation realizations, Florator [5], Weyers [6], and Fhakrabarti *et al* [7] studied the representations of $A(n)_q$. In particular, in a previous paper [8], the authors revealed the structure of $A(n)_q$ and defined their *Verma* modules, and then, on this basis gave a general method for constructing q-boson realizations. In paper [9], the diagonal basis for the Verma module was constructed.

Generalizing our procedure proposed in [11], Burdik *et al* gave recurrent q-boson realizations of quantum enveloping algebra $U_q(sl(n+1,\mathbb{C}))$ [10]. This paper is devoted to the recurrent construction of a q-boson realization of $A(n)_q$.

Here, $\mathbb{C}$ and Z^+ are the field of complex numbers and the set of non-negative integers, respectively. As usual we use abbreviation $[x]=(q^x-q^{-x})/(q-q^{-1})$.

(1) First we recall the structure of $A(n)_q$ and its Verma module. The set $\{K_i=m_{n+1-i\,i}|1\leq i\leq n\}$ generates a maximal commutative subalgebra $\mathcal{H}(n)_q$, which is

called the *Cartan subalgebra*. Elements m_{ij} $(j>n+1-i)$ and m_{ij} $(j<n+1-i)$ are defined as the raising and lowering generators, respectively. For a raising generator m_{ij} $(j>n+1-i)$ there exists a lowering generator $m_{n+1-j\,n+1-i}$ such that

$$[m_{ij}, m_{n+1-j\,n+1-i}] = -(q^{-1}-q)K_jK_{n+1-i} \in \mathscr{H}(n)_q \qquad j>n+1-i. \qquad (3)$$

So we say that $\{m_{ij}, m_{n+1-j\,n+1-i}\}$ $(j>n+1-i)$ is a *pair* which is similar to the pair $\{x_\alpha, y_\alpha\}$ of raising and lowering generators of Lie algebras corresponding to a root α.

The Verma module of $A(n)_q$ is defined as $\mathscr{V}(n)_q \equiv A(n)_q \cdot v_0$, where v_0 is the maximal vector satisfying

$$m_{ij}v_0 = 0 \qquad (j>n+1-i) \qquad K_iv_0 = \lambda_i v_0 \qquad (1 \leqslant i \leqslant n). \qquad (4)$$

It is obvious that $\mathscr{V}(n)_q$ is spanned by

$$\mathscr{V}(n)_q\colon \{X(k_{ij}) \equiv \prod_{ij}{}' (m_{ij})^{k_{ij}} \cdot v_0 \,|\, k_{ij} \in Z^+, j<n+1-i\} \qquad (5)$$

where the notation $'$ means that Π is an ordered product. Then by making use of the procedure we can obtain a q-boson realization of $A(n)_q$ corresponding to the Verma representation (for detail see [11]).

(2) Now we turn to the recurrent construction of the q-boson realization of $A(n)_q$. Dropping the first row and the nth column from $M \in \mathrm{GL}(n)_q$, we obtain an element of $\mathrm{GL}(n-1)_q$. It is worth noting that one has different ways to obtain an element of $\mathrm{GL}(n-1)_q$ from $\mathrm{GL}(n)_q$. However, our choice makes the Cartan, raising, and lowering generators of $\mathrm{GL}(n-1)_q$ remain the Cartan, raising and lowering operators of $\mathrm{GL}(n)_q$, respectively. The matrix elements m_{ij} $(2 \leqslant i \leqslant n, 1 \leqslant j \leqslant n-1)$ of $\mathrm{GL}(n-1)_q$ generate a subalgebra of $A(n)_q$, which is obviously the $A(n-1)_q$.

Let $\bar{A}(n-1)_q$ be a subalgebra of $A(n)_q$ generated by $A(n-1)_q$, the elements m_{in} $(2 \leqslant i \leqslant n)$ and K_n. The left regular representation ρ_{n-1} of $A(n-1)_q$ can be extended to a representation $\bar{\rho}_{n-1}$ of $\bar{A}(n-1)_q$ by

$$\bar{\rho}_{n-1}(x) = \begin{cases} \lambda_n & \text{for } x = K_n \qquad \lambda_n \in \mathbb{C} \\ \rho_{n-1}(x) & \text{for } x \in A(n-1)_q \\ 0 & \text{for } x = m_{in} \qquad (2 \leqslant i \leqslant n). \end{cases} \qquad (6)$$

It is obvious that $A(n)_q \cdot (\bar{A}(n-1)_q \otimes z - 1 \otimes \bar{\rho}_{n-1}(\bar{A}(n-1)_q)z)$, $\forall z \in A(n-1)_q$, is an invariant subspace of the left representation ρ_n of $A(n)_q$. We will denote by τ_n the quotient representation of ρ_n with respect to this invariant subspace.

It is easy to see that the representative space $V(\lambda_n)$ of τ_n is spanned by

$$|k_i> \otimes w \equiv |k_1, k_2, \ldots, k_{n-1}> \otimes w \equiv m_{11}^{k_1} m_{12}^{k_2} \ldots m_{1\,n-1}^{k_{n-1}} \otimes w \qquad k_i \in Z^+ \qquad (7)$$

where $w \in A(n-1)_q$.

By making use of the following relations $(2 \leqslant i \leqslant n)$

$$m_{ij} m_{1t}^{k_t} = \begin{cases} q^{k_t} m_{1t}^{k_t} m_{ij} & \text{for } j=t \\ m_{1t}^{k_t} m_{ij} & \text{for } j<t \\ m_{1t}^{k_t} m_{ij} - q^{-1}(1-q^{2k_t}) m_{1t}^{k_t-1} m_{it} m_{1j} & \text{for } j>t \end{cases}$$

$$m_{1j} m_{1t}^{k_t} = \begin{cases} q^{-k_t} m_{1t}^{k_t} m_{1j} & \text{for } j<t \\ m_{1t}^{k_t} m_{1j} & \text{for } j=t \\ q^{k_t} m_{1t}^{k_t} m_{1j} & \text{for } j>t \end{cases}$$

we can calculate that

$$\tau_n(m_{1j})(|k_t> \otimes w) = q^{\Sigma_{t=1}^{j-1} k_t}(|k_t + \delta_{tj}> \otimes w) \qquad (1 \leqslant j \leqslant n-1)$$

$$\tau_n(m_{1n})(|k_t> \otimes w) = q^{\Sigma_{t=1}^{n-1} k_t} \lambda_n(|k_t> \otimes w)$$

$$\tau_n(m_{in})(|k_t> \otimes w) = -q^{-1} \sum_{r=1}^{n-1} q^{\Delta_{n-r}(r)\, \Sigma_{t=r+1}^{n-1} k_t}(1-q^{2k_r}) \lambda_n(|k_t - \delta_{tr}> \otimes \rho_{n-1}(m_{ir} w)) \qquad (2 \leqslant i \leqslant n)$$

$$\tau_n(m_{ij})(|k_t> \otimes w) = q^{k_j}(|k_t> \otimes \rho_{n-1}(m_{ij} w)) - \sum_{r=1}^{j-1} q^{-1} q^{\Delta_{j-1}(r) \Sigma_{t=r+1}^{j-1} k_t}(1-q^{2k_r})$$
$$\times (|k_t - \delta_{tr} + \delta_{tj}> \otimes \rho_{n-1}(m_{ir} w)) \qquad (2 \leqslant i \leqslant n,\ 2 \leqslant j \leqslant n-1)$$

$$\tau_n(m_{i1})(|k_t> \otimes w) = q^{k_1}(|k_t> \otimes \rho_{n-1}(m_{i1} w)) \qquad (2 \leqslant i \leqslant n) \tag{8}$$

where $\Delta_{j-1}(r) = 1 - \delta_{j-1\,r}$. To obtain the recurrent q-boson realization of $A(n)_q$, we consider the 'generalized' q-Fock representation of q-deformed Heisenberg–Weyl algebras $\mathcal{W}(n-1)_q$ of $n-1$ independent q-bosons on $V(\lambda_n)$

$$b_i^+(|k_t> \otimes w) = |k_t + \delta_{ti}> \otimes w$$

$$b_i(|k_t> \otimes w) = [k_i](|k_t - \delta_{ti}> \otimes w)$$

$$q^{\pm N_i}(|k_t> \otimes w) = q^{\pm k_i}(|k_t> \otimes w). \tag{9}$$

It is easy to check that b_i^+, b_i, $q^{\pm N_i}$ $(1\leqslant i\leqslant n-1)$ satisfy the following defining relations of $\mathcal{W}(n-1)_q$ [12]

$$\begin{aligned} &b_i b_i^+ - q^{\mp 1} b_i^+ b_i = q^{\pm N_i} \\ &q^{N_i} b_i^+ q^{-N_i} = q b_i^+ \qquad q^{N_i} b_i q^{-N_i} = q^{-1} b_i \\ &[x_i, x_j] = 0 \qquad (i\neq j, x_i = b_i^+, b_i, q^{\pm N_i}). \end{aligned} \tag{10}$$

Then from equations (8) it follows that

$$\begin{aligned} &m_{1j} = q^{\sum_{i=1}^{j-1} N_i} b_j^+ \qquad (1\leqslant j\leqslant n-1), \\ &m_{1n} = q^{\sum_{i=1}^{n-1} N_i} \lambda_n \\ &m_{in} = q^{-1} \sum_{r=1}^{n-1} q^{\sum_{i=r}^{n-1} N_i} \lambda_n b_r \rho_{n-1}(m_{ir}) \qquad (2\leqslant i\leqslant n) \\ &m_{ij} = q^{N_j} \rho_{n-1}(m_{ij}) + q^{-1} \sum_{r=1}^{j-1} q^{\sum_{i=r}^{j-1} N_i} b_r b_j^+ \rho_{n-1}(m_{ir}) \qquad (2\leqslant i\leqslant n, 2\leqslant j\leqslant n-1) \\ &m_{i1} = q^{N_1} \rho_{n-1}(m_{i1}) \qquad (2\leq i\leq n) \end{aligned} \tag{11}$$

which is the desired recurrent q-boson realization of $A(n)_q$. We conclude that generators of $A(n)_q$ are expressed as $n-1$ independent q-boson operators and the generators of $A(n-1)_q$.

(3) As a matter of fact, ρ_{n-1} can be chosen as not only the left regular representation of $A(n)_q$, but also representations of $A(n-1)_q$ induced by ρ_{n-1} on some quotient spaces of $A(n-1)_q$ with respect to some invariant subspaces. Extension to $\bar{\rho}_{n-1}$ is in the same way as in equation (6). In particular, if we choose ρ_{n-1} to be the Verma representation, or equivalently, the q-boson realizations corresponding to the Verma representation, the $V(\lambda_n)$ is just the Verma module of $A(n)_q$. In this way we shall obtain a pure q-boson realiztion of $A(n)_q$ corresponding to the Verma representation of $A(n)_q$. In this case the central element $D_q(M)$ will take a constant.

(4) In paper [9], we constructed the so-called diagonal basis for Verma module of $A(n)_q$, on which the Cartan generators K_i are diagonal. Suppose that we have been given the diagonal basis of $A(n-1)_q$. Then, adding the following elements

$$\Delta_1^n = m_{1\,n-1} \qquad \Delta_2^n = D_q \begin{bmatrix} m_{1\,n-2} & m_{1\,n-1} \\ m_{2\,n-2} & m_{2\,n-1} \end{bmatrix}$$

$$\Delta_{n-1}^n = D_q \begin{bmatrix} m_{11} & m_{12} & \dots & m_{1\,n-1} \\ m_{21} & m_{22} & \dots & m_{2\,n-1} \\ \dots & \dots & \dots & \dots \\ m_{n-1\,1} & m_{n-1\,2} & \dots & m_{n-1\,n-1} \end{bmatrix} \tag{12}$$

instead of m_{1i}, we obtain a diagonal basis for $A(n)_q$. In terms of these elements, $V(\lambda_n)$ can also be spanned by

$$(\Delta_1^n)^{k_1}(\Delta_2^n)^{k_2}\dots(\Delta_1^n)^{k_n}\otimes w. \tag{13}$$

Then $\tau_n(K_i)$ will be diagonal on these vectors if $\rho_{n-1}(K_i)$ is diagonal. However, in this case, it is difficult to explicitly calculate the recurrent formulae because the commutation relations between m_{ij} and Δ_i^n are very complicated.

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